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Topics in the Analysis of Shear-Wave Propagation in Oblique-Plate Impact Tests

by Mike Scheidler

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14. ABSTRACT
This report addresses several topics in the theoretical analysis of shock waves, acceleration waves, and centered simple waves, with emphasis on the propagation of shear waves generated in oblique-plate impact tests. The first report, “Formulas for the Pressure and Bulk Modulus in Uniaxial Strain,” demonstrates that for a general, nonlinear, isotropic elastic solid, the shear stress in the uniaxially strained region ahead of the shear wave may be inferred from the measured shear-wave speed and the density. This result, which improves on approximate analyses in the literature, is applied to the study of fused silica in the shocked state. The second report, “Response of Nonlinear Elastic Solids to Oblique-Plate Impact,” treats the case where the shear wave is a centered simple wave (or ramp wave). Approximate relations are derived for the stress, strain, particle velocity, wave speed, and rise time, and applications to material characterization are discussed. The third report, “Universal Relations for Pressure-Shear Waves in Nonlinear Elastic Solids,” treats centered simple shear waves and shear shocks. Here, we obtain approximate relations between stress, strain, particle velocity, and wave speeds that do not explicitly involve the constitutive functions or elastic constants of the material. The fourth report, “Approximate Universal Relations Between Shock and Acceleration Wave Speeds for Oblique Plate Impact of Inelastic Solids,” extends some of the results of the previous report to a large class of inelastic solids, although a completely different method of proof is required for the inelastic case. The results of the latter three reports apply not only to isotropic solids but also to certain anisotropic solids, provided that the symmetry axes are appropriately aligned with the test geometry.

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Preface

This report contains reprints of four reports[1–4] that focus on various theoretical aspects of the propagation of shock waves and acceleration waves in nonlinear elastic or inelastic solids. The common theme of these reports is the analysis of the shear waves generated in an oblique-plate impact test. The papers appeared in different years in the Proceedings of the Conference of the American Physical Society Topical Group on Shock Compression of Condensed Matter.

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[1] Scheidler, M. Formulas for the Pressure and Bulk Modulus in Uniaxial Strain. In *Shock Compression of Condensed Matter – 1995;* Schmidt, S. C., Tao, W. C., Eds.; American Institute of Physics: Melville, NY, 1996; pp 475–478.

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[3] Scheidler, M. Universal Relations for Pressure-Shear Waves in Nonlinear Elastic Solids. In *Shock Compression of Condensed Matter – 1997;* Furnish, M. D., Chhabildas, L. C., Hixson, R. S., Eds.; American Institute of Physics: Melville, NY, 2000; pp 181–184.

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FORMULAS FOR THE PRESSURE AND BULK MODULUS IN UNIAXIAL STRAIN

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For an isotropic elastic solid, the pressure $p = p_u(\rho)$ in a state of uniaxial strain at density ρ generally differs from the pressure $p = p_h(\rho)$ in a state of hydrostatic stress at the same density. Several researchers have used pressure/shear (or oblique plate impact) tests to determine p_u and the corresponding uniaxial bulk modulus $K_u \equiv \rho\, dp_u/d\rho$. The pressure/shear tests yield uniaxial longitudinal and shear moduli, L_u and G_u, as a function ρ. A common procedure is to integrate the approximate relation $K_u \approx L_u - \frac{4}{3}G_u$ to obtain the pressure-density relation $p = p_u(\rho)$ in uniaxial strain. It is shown here that the integration of this *approximate* relation between the moduli can be avoided altogether by utilizing the *exact* formula $p_u = \sigma_1 - \frac{2}{3}((\rho/\rho_0)^2 - 1)G_u$, where σ_1 denotes the longitudinal stress (pos. in compression). The bulk modulus K_u is computed exactly from this formula, and the error in approximating it by $L_u - \frac{4}{3}G_u$ is determined.

1. INTRODUCTION

We consider only isotropic elastic materials, and for simplicity thermal effects are neglected until §5. Under these conditions, the pressure is typically assumed to be a function of density only. However, nonlinear elasticity theory predicts that the pressure also depends on the shear strain, although isotropy implies this effect is necessarily of second order; cf. Scheidler (1). In §4 we derive exact formulas for the pressure and bulk modulus in a state of uniaxial strain. The effect of shear strain can be seen by comparing these results with the corresponding relations for a state of hydrostatic stress (§3). Our results are based on exact formulas for the speeds of acceleration waves (§2). Applications to the analysis of data from pressure/shear tests are discussed in §5.

2. ACCELERATION WAVE SPEEDS

Let $\mathbf{F}$ denote the deformation gradient relative to the undeformed and unstressed state. The left Cauchy-Green tensor $\mathbf{B} \equiv \mathbf{F}\mathbf{F}^\mathsf{T}$ has principal values $b_i = \lambda_i^2$, where λ_i are the principal stretches, and

$$\det \mathbf{F} = \sqrt{b_1 b_2 b_3} = \lambda_1\lambda_2\lambda_3 = \frac{1}{\tilde{\rho}}, \quad \tilde{\rho} \equiv \frac{\rho}{\rho_0}, \quad (1)$$

where ρ and ρ_0 denote the densities in the deformed and undeformed state. The principal axes of $\mathbf{B}$ are the principal axes of strain in the deformed state. Since the material is isotropic and elastic, these axes are also the principal axes of the Cauchy stress tensor $\mathbf{T}$, and $\mathbf{T}$ is an isotropic function of $\mathbf{B}$. This implies that there is a single function $\hat{t}$ such that the principal stresses t_i are given by

$$t_i = \hat{t}(b_i, b_j, b_k) = \hat{t}(b_i, b_k, b_j), \quad (2)$$

for any permutation i, j, k of $1, 2, 3$; cf. Truesdell & Noll (2). It follows that the pressure

$$p \equiv -\tfrac{1}{3}\mathrm{tr}\,\mathbf{T} = -\tfrac{1}{3}(t_1 + t_2 + t_3) \quad (3)$$

is a symmetric function of b_1, b_2, b_3. Analogous results hold in terms of the principal stretches λ_i or in terms of various principal strain measures, e.g., $\lambda_i - 1$, $\frac{1}{2}(b_i - 1)$, $\frac{1}{2}(1 - 1/b_i)$, or $\ln \lambda_i$.

The speed U_i of a longitudinal acceleration wave propagating along the ith principal axis of strain is given by

$$\rho\, U_i^2 = 2b_i \frac{\partial t_i}{\partial b_i} = \lambda_i \frac{\partial t_i}{\partial \lambda_i} = \frac{\partial t_i}{\partial \ln \lambda_i}. \quad (4)$$

The speed U_{ij} of a transverse or shear acceleration wave propagating along the ith principal axis

of strain with jump in acceleration parallel to the jth principal axis ($j \neq i$) is given by

$$\begin{aligned} \rho U_{ij}^2 &= b_i \left(\frac{\partial t_i}{\partial b_i} - \frac{\partial t_i}{\partial b_j} \right), \quad \text{if} \quad b_i = b_j, \\ &= b_i \frac{t_i - t_j}{b_i - b_j}, \quad \text{if} \quad b_i \neq b_j. \end{aligned} \tag{5}$$

All quantities in (4) and (5) are evaluated at the wave front. These wave speeds are in the deformed material (i.e., Eulerian); the corresponding Lagrangian wave speeds are obtained by dividing by λ_i. Proofs of (4) and (5) can be found in Truesdell & Noll (2) and Wang & Truesdell (3). These formulae do not require that the region ahead of the wave be at rest or in a homogeneous state of strain. However, when these conditions are satisfied, (4) and (5) also apply to the speeds of plane infinitesimal sinusoidal waves; cf. Truesdell & Noll (2).

3. HYDROSTATIC STRESS

For a purely dilatational deformation, we have

$$b_i = \tilde{\rho}^{-2/3} \quad \text{and} \quad t_i = -p \quad (i = 1, 2, 3). \tag{6}$$

In this hydrostatic stress state every axis is a principal axis of stress and strain, and the pressure p is a function p_h of ρ or $\tilde{\rho}$. Here and below, an "h" subscript denotes the hydrostatic stress state. From (2), (4), $(5)_1$, and (6), it follows that for a given density ρ there is a single longitudinal wave speed $U_i = U_{L,h}$ and a single shear wave speed $U_{ij} = U_{S,h}$, and that

$$\frac{dp_h}{d\rho} = U_{L,h}^2 - \tfrac{4}{3} U_{S,h}^2 ; \tag{7}$$

cf. Wang & Truesdell (3). A different proof of this well-known result is given by Truesdell & Noll (2). We assume p_h is a strictly increasing function of ρ. Then (7) implies the longitudinal wave speed is greater than the shear wave speed. With the longitudinal, shear, and bulk moduli defined by

$$L_h \equiv \rho U_{L,h}^2 \qquad G_h \equiv \rho U_{S,h}^2 \tag{8}$$

$$K_h \equiv \rho \frac{dp_h}{d\rho} = \tilde{\rho} \frac{dp_h}{d\tilde{\rho}}, \tag{9}$$

(7) implies the well-known relation

$$K_h = L_h - \tfrac{4}{3} G_h . \tag{10}$$

We use a zero subscript to denote functions evaluated at the undeformed state where $\lambda_i = b_i = \tilde{\rho} = 1$; in particular, $K_0 = L_0 - \frac{4}{3} G_0$. By (9),

$$p_h / K_0 \approx (\tilde{\rho} - 1) + a_0 (\tilde{\rho} - 1)^2, \tag{11}$$

where the dimensionless constant a_0 is given by

$$a_0 = \frac{1}{2K_0} \left. \frac{d^2 p_h}{d\tilde{\rho}^2} \right|_0 = \frac{1}{2} \left(\left. \frac{dK_h}{dp_h} \right|_0 - 1 \right). \tag{12}$$

4. UNIAXIAL STRAIN

For a state of uniaxial strain along the 1-axis,

$$\lambda_1 = \sqrt{b_1} = 1/\tilde{\rho}, \quad \lambda_2 = \lambda_3 = b_2 = b_3 = 1 \tag{13}$$

and (2) implies $t_2 = t_3$. The principal stresses t_i are positive in tension; if $\sigma_i \equiv -t_i$ then σ_i is positive in compression. We use a "u" subscript to denote uniaxial strain and consider only waves propagating along the 1-axis into a uniaxially strained material. The Eulerian wave speed $U_{L,u} = U_1$ of a longitudinal acceleration wave is given by (4) with $i = 1$, and by (13) we also have

$$L_u \equiv \rho U_{L,u}^2 = \tilde{\rho} \frac{d\sigma_1}{d\tilde{\rho}} = \rho \frac{d\sigma_1}{d\rho} \tag{14}$$

It follows that a longitudinal acceleration wave can propagate only if $d\sigma_1 / d\tilde{\rho} > 0$, i.e., if σ_1 is a strictly increasing function of $\tilde{\rho}$, which we now assume. By (13), the material is strained iff $\tilde{\rho} \neq 1$ iff $b_1 \neq b_2$. In this case $(5)_2$ and (13) imply the following formulas for the Eulerian speed $U_{S,u} = U_{12}$ of a transverse or shear acceleration wave:

$$\begin{aligned} G_u \equiv \rho U_{S,u}^2 &= b_1 \frac{t_1 - t_2}{b_1 - b_2} = \frac{t_1 - t_2}{1 - \tilde{\rho}^2} \\ &= \frac{\sigma_1 - \sigma_2}{\tilde{\rho}^2 - 1} = \frac{2\tau}{\tilde{\rho}^2 - 1}, \end{aligned} \tag{15}$$

where τ is the shear stress:

$$\tau \equiv \tfrac{1}{2}(\sigma_1 - \sigma_2) = \tfrac{1}{2}(t_2 - t_1). \tag{16}$$

The Lagrangian wave speeds are $\tilde{\rho}\, U_{L,u}$ and $\tilde{\rho}\, U_{S,u}$. If $\tilde{\rho} > 1$ the material is in compression, and (15)

implies that a shear acceleration wave can propagate only if $\sigma_1 > \sigma_2$. When $\tilde{\rho} = 1$, the results of the previous section apply, and we have $G_u|_0 = G_0$ and $L_u|_0 = L_0$.

From (15) we have the following fundamental formula for the shear stress in uniaxial strain:

$$\tau = \tfrac{1}{2}(\tilde{\rho}^2 - 1)G_u . \qquad (17)$$

Since $t_2 = t_3$, (3) and (16) imply the following well-known relation between the (compressive) longitudinal stress σ_1, the shear stress τ, and the pressure p_u in uniaxial strain:

$$p_u = \sigma_1 - \tfrac{4}{3}\tau . \qquad (18)$$

On substituting (17) into (18), we obtain the following fundamental formula for p_u:

$$p_u = \sigma_1 - \tfrac{2}{3}(\tilde{\rho}^2 - 1)G_u . \qquad (19)$$

We define the bulk modulus K_u in uniaxial strain by

$$K_u \equiv \rho \frac{dp_u}{d\rho} = \tilde{\rho}\frac{dp_u}{d\tilde{\rho}} = L_u \frac{dp_u}{d\sigma_1} . \qquad (20)$$

Then from (19) and (14), we obtain

$$\begin{aligned} K_u &= L_u - \tfrac{4}{3}\tilde{\rho}^2 G_u - \tfrac{2}{3}(\tilde{\rho}^2 - 1)\tilde{\rho}\frac{dG_u}{d\tilde{\rho}} \\ &= (L_u - \tfrac{4}{3}G_u) - \tfrac{2}{3}(\tilde{\rho}^2 - 1)H_u , \end{aligned} \qquad (21)$$

$$H_u = 2G_u + \tilde{\rho}\frac{dG_u}{d\tilde{\rho}} = \frac{1}{\tilde{\rho}}\frac{d}{d\tilde{\rho}}(\tilde{\rho}^2 G_u) . \qquad (22)$$

At $\tilde{\rho} = 1$, (21) reduces to $K_u|_0 = L_0 - \frac{4}{3}G_0 = K_0$. From (21) and (22) it follows that $K_u = L_u - \frac{4}{3}G_u$ for all $\tilde{\rho}$ iff $H_u = 0$ iff $G_u = G_0/\tilde{\rho}^2$, but there is no reason to expect such dependence in general, and thus no reason to expect that $K_u = L_u - \frac{4}{3}G_u$ except in the limit of zero strain. Of course, by analogy with (10) we could have defined K_u to be $L_u - \frac{4}{3}G_u$, but then (20) would not hold. From (21) we see that for a state of compression, $K_u < L_u - \frac{4}{3}G_u$ if $H_u > 0$, and $K_u > L_u - \frac{4}{3}G_u$ if $H_u < 0$. We assume that p_u is a strictly increasing function of $\tilde{\rho}$. Then any function of $\tilde{\rho}$ may also be regarded as a function of σ_1 or p_u, and by (14) and (20) we have

$$\rho\frac{d}{d\rho} = \tilde{\rho}\frac{d}{d\tilde{\rho}} = L_u\frac{d}{d\sigma_1} = K_u\frac{d}{dp_u} . \qquad (23)$$

The results up to this point are exact. We now consider some useful approximate relations. From (20) we have

$$p_u/K_0 \approx (\tilde{\rho} - 1) + b_0(\tilde{\rho} - 1)^2 , \qquad (24)$$

where the dimensionless constant b_0 is given by

$$b_0 = \frac{1}{2K_0}\left.\frac{d^2 p_u}{d\tilde{\rho}^2}\right|_0 = \frac{1}{2}\left(\left.\frac{dK_u}{dp_u}\right|_0 - 1\right) . \qquad (25)$$

For use in (24)–(25), note that (21) implies

$$\left.\frac{dK_u}{dp_u}\right|_0 = \frac{L_0}{K_0}\left(\left.\frac{dL_u}{d\sigma_1}\right|_0 - \frac{8}{3}\left.\frac{dG_u}{d\sigma_1}\right|_0\right) - \frac{8}{3}\frac{G_0}{K_0} . \qquad (26)$$

From (11)–(12) and (24)–(25), we see that

$$p_u \approx p_h + K_0 c_0(\tilde{\rho} - 1)^2 , \qquad (27)$$

$$\frac{p_u - p_h}{p_u} \approx \frac{p_u - p_h}{p_h} \approx c_0(\tilde{\rho} - 1) , \qquad (28)$$

where the dimensionless constant c_0 is given by

$$c_0 = b_0 - a_0 = \frac{1}{2}\left(\left.\frac{dK_u}{dp_u}\right|_0 - \left.\frac{dK_h}{dp_h}\right|_0\right) . \qquad (29)$$

On comparing (28) with equation (4.6) in Scheidler (1), we find that c_0 is also given by

$$c_0 = \frac{2}{3}\left(\left.\frac{dG_h}{dp_h}\right|_0 - \frac{G_0}{K_0}\right) . \qquad (30)$$

6. DISCUSSION

The longitudinal stress σ_1 as a function of $\tilde{\rho}$ in uniaxial strain can be obtained from normal plate impact tests. Then the relation (18) (which does not rely on the assumption that the response is elastic) is typically used to determine the pressure p_u in uniaxial strain given some assumptions on the shear stress τ, or to determine τ given some assumptions on p_u. It is often assumed that $p_u(\tilde{\rho})$ is equal to the pressure $p_h(\tilde{\rho})$ in a state of hydrostatic stress at density $\tilde{\rho}$ (or to some appropriate modification of p_h to include thermal effects in the shocked state). Such an approximation neglects the effects of shear strain (or shear stress) on p_u. That this effect may be significant in ceramics, geologic materials, and polymers has

been emphasized by Gupta (4) and Conner (5). These materials can sustain relatively large elastic shear strains (compared to metals), although for polymers viscoelastic effects should also be taken into account. Only elastic response is considered here. Then (27) implies that $p_u(\tilde{\rho})$ differs from $p_h(\tilde{\rho})$ by a term of order $(\tilde{\rho}-1)^2$ unless $c_0 = 0$, which is generally not the case. If c_0 and $p_h(\tilde{\rho})$ are known, then (27) provides an approximation to p_u to within an error of order $(\tilde{\rho}-1)^3$. The relative error in approximating p_u by p_h is of order $\tilde{\rho}-1$ and can be estimated by using (28).

In a pressure/shear (or oblique plate impact) test, a longitudinal wave propagates into the undeformed material bringing it to a state of uniaxial strain, and a slower shear wave propagates into this uniaxially strained material. These tests yield both $\sigma_1(\tilde{\rho})$ and the shear wave speed $U_{S,u}$ (and hence G_u) as a function of $\tilde{\rho}$ or σ_1. If the shear wave travels at the acceleration wave speed then (17), (19), and (21) provide *exact* formulas for the shear stress τ, the pressure p_u, and the bulk modulus K_u in uniaxial strain as a function of $\tilde{\rho}$ or σ_1. These formulas appear to have gone unnoticed, however. Instead, it is usually assumed that $K_u \approx L_u - \frac{4}{3}G_u$. This approximate relation, together with (20), is then integrated to give p_u as a function of $\tilde{\rho}$; cf. Gupta (4,6), Conner (5), and Aidun & Gupta (7). For fused silica in the strain range $0 \leq \tilde{\rho}-1 \leq 0.076$, the response is elastic and the shear wave speed decreases with $\tilde{\rho}$; cf. Conner (5). In this strain range the shear wave is an acceleration wave (cf. also Abou-Sayed & Clifton (8)), so we may apply the results of §4. Using (21) and Conner's data, we find that at a strain of $\tilde{\rho}-1 = 0.076$ the estimate $K_u \approx L_u - \frac{4}{3}G_u$ is low by about 29%.

Whether the shear wave in a pressure/shear test is an acceleration wave or a shock wave depends on the nonlinear elastic response of the material; cf. Davison (9). The shear modulus G_u in §4 is defined in terms of the acceleration wave speed $U_{S,u}$, or equivalently, in terms of the speed of a plane infinitesimal sinusoidal shear wave; cf. §2. If a shear shock with speed $\bar{U}$ can propagate in the uniaxially strained material and if we set $\bar{G} \equiv \rho\bar{U}^2$, then the formulas in §4 hold approximately when G_u is replaced with $\bar{G}$. Also note that if $\bar{U} > U_{S,u}$ (as standard stability arguments would imply), then $\bar{G} > G_u$, and (17) and (19) imply that $\tau < \frac{1}{2}(\tilde{\rho}^2-1)\bar{G}$ and $p_u > \sigma_1 - \frac{2}{3}(\tilde{\rho}^2-1)\bar{G}$ in compression ($\tilde{\rho} > 1$).

We conclude with a brief discussion of thermodynamic effects, which have been neglected up to this point. If a thermoelastic material conducts heat by Fourier's law [resp. is a nonconductor], then a longitudinal acceleration wave propagates at the isothermal [resp. adiabatic] wave speed. However, the formula (15) for the speed of a shear acceleration continues to hold in either case; cf. Bowen & Wang (10). In fact, it can be shown that (15) holds even if heat conduction is governed by Cattaneo's equation, which prohibits instantaneous propagation of thermal disturbances. Thus the formulas (17) and (19) for the shear stress and the pressure continue to hold. In particular, they are valid when the state of uniaxial strain has been achieved by shock loading.

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RESPONSE OF NONLINEAR ELASTIC SOLIDS TO OBLIQUE PLATE IMPACT

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We give a theoretical analysis of the nonlinear elastic response in the interior of the target in an oblique plate impact test, prior to interactions with any reflected waves. Approximate relations are derived between the changes in stress, strain, particle velocity, and wave speeds across the shear wave by neglecting shear strain terms of order six in the internal energy function. The results are valid for any finite (elastic) longitudinal strain ahead of the shear wave. They apply to isotropic materials and to appropriately aligned transversely isotropic and orthotropic materials.

INTRODUCTION

In an oblique plate impact test a flyer plate, inclined relative to the axis of the projectile, impacts a parallel target plate, which we assume is at rest and stress free. Let (X, Y, Z) denote the Cartesian coordinates of a material point in this natural reference state, with the X-axis normal to the target face. The particle velocity $\mathbf{v}_F$ imparted to the face of the target has nonzero components both normal and parallel to the target face. The coordinate axes are oriented so that $\mathbf{v}_F = (u_F, v_F, 0)$, with the normal ($X$) component $u_F > 0$ and the transverse (Y) component $v_F > 0$. The impact is assumed weak enough that the target response is elastic. We assume the target is homogeneous, and restrict attention to points in the interior and times prior to the arrival of waves reflected from free surfaces or material interfaces. Then the motion is independent of the Y and Z coordinates. However, for a general anisotropic material, the motion may have nonzero components in the Z-direction at points not on the impact face. For isotropic materials as well as some anisotropic materials (appropriately aligned with the coordinate axes), there is no motion in the Z-direction. This is the case considered here.

PRELIMINARIES

If (x, y, z) is the position at time t of the material point initially at (X, Y, Z), then $x = X + d_1(X,t)$, $y = Y + d_2(X,t)$, $z = Z$. The normal and transverse components of the particle velocity are $u = \partial x/\partial t$ and $v = \partial y/\partial t$. Let $\boldsymbol{P}$ and $\boldsymbol{T}$ denote the 1st Piola-Kirchhoff and Cauchy stress tensors. Then $\sigma \equiv -P_{XX} = -T_{XX}$ and $\varepsilon \equiv 1 - \partial x/\partial X$ are the normal or longitudinal components of stress and strain, taken positive in compression; and $\tau \equiv -P_{YX} = -T_{YX}$ and $\gamma \equiv -\partial y/\partial X$ are the shear stress and shear strain. The signs have been chosen so that ε, γ, σ, τ, as well as u and v, should be nonnegative for the impact problem considered here. Since ε and γ are the only nonzero strain components, the internal energy e per unit mass depends only on ε, γ, and the entropy per unit mass s. Then

$$e = \hat{e}(\varepsilon, \gamma, s)\,, \quad \sigma = \rho_0 \frac{\partial e}{\partial \varepsilon}\,, \quad \tau = \rho_0 \frac{\partial e}{\partial \gamma}\,, \quad (1)$$

where ρ_0 is the density in the natural state. Heat conduction is neglected in the analysis, so that response is isentropic except for entropy jumps across shocks. The equations of motion are

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial \sigma}{\partial X} = 0\,, \quad \rho_0 \frac{\partial v}{\partial t} + \frac{\partial \tau}{\partial X} = 0\,. \quad (2)$$

The strain rate and velocity gradient satisfy

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial u}{\partial X} = 0\,, \quad \frac{\partial \gamma}{\partial t} + \frac{\partial v}{\partial X} = 0\,. \qquad (3)$$

Smooth solutions are governed by (2) and (3), with σ and τ given by (1) and s constant; this is a quasilinear system in the four unknowns u, v, ε, γ. The (Lagrangian) characteristic wavespeeds are the roots $\pm U$ and $\pm V$ of

$$2\rho_0 \begin{Bmatrix} U^2 \\ V^2 \end{Bmatrix} = \frac{\partial \sigma}{\partial \varepsilon} + \frac{\partial \tau}{\partial \gamma} \pm \sqrt{\left(\frac{\partial \sigma}{\partial \varepsilon} - \frac{\partial \tau}{\partial \gamma}\right)^2 + 4\left(\frac{\partial \tau}{\partial \varepsilon}\right)^2}\,, \qquad (4)$$

where the fast wave speeds $\pm U$ correspond to the $+$ sign and the slow wave speeds $\pm V$ to the $-$ sign in (4). The constitutive inequalities

$$\frac{\partial \sigma}{\partial \varepsilon} > \frac{\partial \tau}{\partial \gamma} > 0\,, \quad \frac{\partial \sigma}{\partial \varepsilon}\frac{\partial \tau}{\partial \gamma} > \left(\frac{\partial \tau}{\partial \varepsilon}\right)^2 \qquad (5)$$

guarantee that there are four real distinct characteristic wave speeds (i.e., $U > V > 0$), and that the fast waves are primarily longitudinal. Waves with negative speed would be generated by reflections and hence are not considered here. The right inequality in (5) is equivalent to

$$\frac{\partial^2 e}{\partial \varepsilon^2}\frac{\partial^2 e}{\partial \gamma^2} > \left(\frac{\partial^2 e}{\partial \varepsilon \partial \gamma}\right)^2\,, \qquad (6)$$

which implies that e is a strictly convex function of ε and γ.

The one-dimensional plane waves for this oblique impact problem are centered simple waves and/or centered shocks. For nonlinear isotropic elastic solids, some of the earlier papers on this problem are Bland (1), Davison (2), and Abou-Sayed & Clifton (3). Isotropy implies that e is an even function of γ for fixed ε and s, and that the fast wave is purely longitudinal, i.e., there are no changes in v, γ, or τ. The fast wave brings the material to an intermediate state of uniform uniaxial strain. The slow wave propagates into this uniaxially strained material. The slow wave is primarily a transverse or shear wave. However, nonlinear elastic effects cause 2nd order changes in u, ε, and σ across the shear wave. The case where both waves are simple waves is illustrated in Fig. 1.

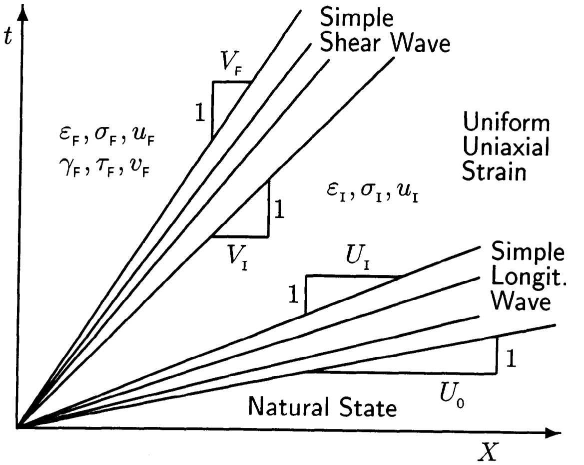

FIGURE 1. Simple wave solutions for oblique plate impact. Quanties in th4e intermediate state (ahead of the shear wave) and final state (behind the shear wave) were denoted by I and F subscriptes, respectively.

For anisotropic materials there may be motion in the Z direction. Even if there is no motion in the Z direction, the fast wave need not be purely longitudinal. However, if the material is orthotropic relative to the X, Y, Z axes, or transversely isotropic about the X or Z axis, then it can be shown that there is no motion in the Z direction, and that e is an even function of γ. Furthermore, it can be shown that this last property implies all of the results discussed previously for the isotropic case. With these restrictions, most of the analysis for isotropic materials in references (1)–(3) can be extended to the anisotropic case. Since e is an even function of γ for fixed ε and s,

$$e \overset{6}{\approx} \hat{f}(\varepsilon, s) + \tfrac{1}{2}\,\hat{g}(\varepsilon, s)\,\gamma^2 + \tfrac{1}{4!}\,\hat{h}(\varepsilon, s)\,\gamma^4. \quad (7)$$

Here and below, $a \overset{n}{\approx} b$ means $a = b$ to within an error of order γ^n. Note that no approximations are made for the dependence of e on ε or s. In particular, the results in this paper are valid for arbitrary finite (elastic) longitudinal strains ahead of the shear wave. We study only the shear wave in this paper, and consider only the case where it is a simple wave (as opposed to a shock). s_0 and s_I denote the entropy in the natural and intermediate states . If the longitudinal wave is simple, $s_I = s_0$; if it is a shock, $s_I > s_0$.

SIMPLE SHEAR WAVES

Set $f(\varepsilon) = \hat{f}(\varepsilon, s_{\mathrm{I}})$, $g(\varepsilon) = \hat{g}(\varepsilon, s_{\mathrm{I}})$, and $h(\varepsilon) = \hat{h}(\varepsilon, s_{\mathrm{I}})$. Since the deformation in the simple shear wave is isentropic, (7), (1), and (4) imply

$$e \overset{6}{\approx} f(\varepsilon) + \tfrac{1}{2}\, g(\varepsilon)\,\gamma^2 + \tfrac{1}{4!}\, h(\varepsilon)\,\gamma^4, \tag{8}$$

$$\frac{\sigma}{\rho_0} = \frac{\partial e}{\partial \varepsilon} \overset{4}{\approx} f'(\varepsilon) + \tfrac{1}{2}\, g'(\varepsilon)\,\gamma^2, \tag{9}$$

$$\frac{\tau}{\rho_0} = \frac{\partial e}{\partial \gamma} \overset{5}{\approx} g(\varepsilon)\,\gamma + \tfrac{1}{6}\, h(\varepsilon)\,\gamma^3, \tag{10}$$

$$V^2 \overset{4}{\approx} g(\varepsilon) - \left(\frac{(g')^2}{f'' - g} - \frac{h}{2}\right)(\varepsilon)\,\gamma^2, \tag{11}$$

where a prime denotes differentiation with respect to ε. Note that σ and V are even functions of γ for fixed ε, while τ is an odd function of γ.

In the intermediate uniaxially strained state ahead of the shear wave, $\gamma = \tau = v = 0$, and by (9)–(11), (4), and $(5)_1$, $\sigma_{\mathrm{I}} = \rho_0 f'(\varepsilon_{\mathrm{I}})$ and

$$U_{\mathrm{I}}^2 = f''(\varepsilon_{\mathrm{I}}) > g(\varepsilon_{\mathrm{I}}) = V_{\mathrm{I}}^2. \tag{12}$$

The rate of change of the speed V_{I} of the shear wave front (which is a transverse acceleration wave) with respect to the uniaxial strain ahead of the wave is

$$V_{\mathrm{I}}' \equiv \frac{dV_{\mathrm{I}}}{d\varepsilon_{\mathrm{I}}} = \frac{g'(\varepsilon_{\mathrm{I}})}{2\sqrt{g(\varepsilon_{\mathrm{I}})}}. \tag{13}$$

In the centered simple shear wave, $V = X/t$; and ε, σ, u, γ, τ & v are constant along the straight-line characteristics (i.e., functions of X/t only). The characteristic form of the quasilinear system yields the differential relations

$$\frac{du}{dv} = \frac{d\sigma}{d\tau} = \frac{d\varepsilon}{d\gamma} = \frac{-\dfrac{\partial \tau}{\partial \varepsilon}}{\dfrac{\partial \sigma}{\partial \varepsilon} - \rho_0 V^2}, \tag{14}$$

$$\frac{dv}{d\gamma} = V, \quad \frac{d\tau}{dv} = \rho_0 V, \quad \frac{d\tau}{d\gamma} = \rho_0 V^2. \tag{15}$$

γ, τ & v are strictly increasing with passage of the wave, and any two of these variables is an odd function of the other. ε, σ, u & V are even functions of any one of the variables γ, τ or v; hence the total derivatives in (14) are 0 at the wavefront, where $\gamma = \tau = v = 0$. Indeed, $\partial\sigma/\partial\varepsilon - \rho_0 V^2 > 0$ by $(5)_1$, while by (10), $\partial\tau/\partial\varepsilon = 0$ at the wavefront. If $\partial\tau/\partial\varepsilon \neq 0$ for $\gamma \neq 0$, then (15) holds with $v, \gamma \,\&\, \tau$ replaced by $u, \varepsilon \,\&\, \sigma$, respectively. Inequality $(5)_3$ places no restriction on the sign of $\partial\tau/\partial\varepsilon$; if it changes sign within the shear wave, then by (14), u, σ & ε do not vary monotonically through the wave.

Next, we quantify the above statements by deriving approximate relations which hold throughout the shear wave. On using (9)–(13) in $(14)_3$ and noting that $\varepsilon = \varepsilon_{\mathrm{I}}$ when $\gamma = 0$, we obtain

$$\varepsilon \overset{4}{\approx} \varepsilon_{\mathrm{I}} - \tfrac{1}{2}\, Q(\varepsilon_{\mathrm{I}})\,\gamma^2, \tag{16}$$

$$Q(\varepsilon_{\mathrm{I}}) \equiv \frac{g'}{f'' - g}(\varepsilon_{\mathrm{I}}) = \frac{2\, V_{\mathrm{I}} V_{\mathrm{I}}'}{U_{\mathrm{I}}^2 - V_{\mathrm{I}}^2}. \tag{17}$$

On substituting (16) into (11) and using (12), (13), (17), and $V = V_{\mathrm{I}}$ when $\gamma = 0$, we obtain

$$V^2 \overset{4}{\approx} V_{\mathrm{I}}^2 - R(\varepsilon_{\mathrm{I}})\,\gamma^2, \quad V \overset{4}{\approx} V_{\mathrm{I}} - \frac{R(\varepsilon_{\mathrm{I}})}{2\, V_{\mathrm{I}}}\,\gamma^2, \tag{18}$$

$$\begin{aligned} 2\, R(\varepsilon_{\mathrm{I}}) &\equiv \frac{3\,(g')^2}{f'' - g}\,(\varepsilon_{\mathrm{I}}) - h(\varepsilon_{\mathrm{I}}) \\ &= \frac{12\, V_{\mathrm{I}}^2}{U_{\mathrm{I}}^2 - V_{\mathrm{I}}^2}\,(V_{\mathrm{I}}')^2 - h(\varepsilon_{\mathrm{I}}). \end{aligned} \tag{19}$$

From $(15)_1$, $(18)_2$, and $v = 0$ when $\gamma = 0$, we get

$$v \overset{5}{\approx} V_{\mathrm{I}}\,\gamma - \frac{R(\varepsilon_{\mathrm{I}})}{6\, V_{\mathrm{I}}}\,\gamma^3, \tag{20}$$

which may be inverted to give

$$\gamma \overset{5}{\approx} \frac{v}{V_{\mathrm{I}}} + \frac{R(\varepsilon_{\mathrm{I}})}{6\, V_{\mathrm{I}}^2}\left(\frac{v}{V_{\mathrm{I}}}\right)^3. \tag{21}$$

Then $(18)_2$, (21), and $(15)_2$ imply

$$V \overset{4}{\approx} V_{\mathrm{I}} - \frac{R(\varepsilon_{\mathrm{I}})}{2\, V_{\mathrm{I}}}\left(\frac{v}{V_{\mathrm{I}}}\right)^2, \tag{22}$$

$$\frac{\tau}{\rho_0} \overset{5}{\approx} V_{\mathrm{I}}\, v - \frac{R(\varepsilon_{\mathrm{I}})}{6\, V_{\mathrm{I}}^3}\, v^3. \tag{23}$$

Similar arguments yield the relations

$$\varepsilon \overset{4}{\approx} \varepsilon_{\mathrm{I}} - \tfrac{1}{2}\, Q(\varepsilon_{\mathrm{I}})\,(v/V_{\mathrm{I}})^2, \tag{24}$$

$$\sigma \overset{4}{\approx} \sigma_{\mathrm{I}} - \tfrac{1}{2}\, Q(\varepsilon_{\mathrm{I}})\,\rho_0 v^2, \tag{25}$$

$$u \overset{4}{\approx} u_{\mathrm{I}} - \tfrac{1}{2}\, Q(\varepsilon_{\mathrm{I}})\, v^2/V_{\mathrm{I}}. \tag{26}$$

Dropping the v^3 terms in (21) and (23) and eliminating $Q(\varepsilon_{\mathrm{I}})$ from (24)–(26) yields

$$\gamma \overset{3}{\approx} v/V_{\mathrm{I}}\,, \qquad \varepsilon - \varepsilon_{\mathrm{I}} \overset{4}{\approx} (u - u_{\mathrm{I}})/V_{\mathrm{I}}\,, \quad (27)$$

$$\tau \overset{3}{\approx} \rho_0 V_{\mathrm{I}} v\,, \qquad \sigma - \sigma_{\mathrm{I}} \overset{4}{\approx} \rho_0 V_{\mathrm{I}}(u - u_{\mathrm{I}})\,. \quad (28)$$

More accurate material-independent approximations for γ and τ can be obtained by solving (22) for $R(\varepsilon_{\mathrm{I}})$ and substituting into (21) and (23):

$$\gamma \overset{5}{\approx} [\,1 + \tfrac{1}{3}(V_{\mathrm{I}}/V_{\mathrm{F}} - 1)(v/v_{\mathrm{F}})^2\,]\, v/V_{\mathrm{I}}, \quad (29)$$

$$\tau \overset{5}{\approx} \rho_0 [\,V_{\mathrm{I}} - \tfrac{1}{3}(V_{\mathrm{I}} - V_{\mathrm{F}})(v/v_{\mathrm{F}})^2\,]\, v. \quad (30)$$

Since V must decrease with passage of the wave, (18) implies that $R(\varepsilon_{\mathrm{I}}) \geq 0$ is necessary for the centered shear wave to be simple (as opposed to a shock) for sufficiently weak impacts, while $R(\varepsilon_{\mathrm{I}}) > 0$ is sufficient. It can be shown that $R(\varepsilon_{\mathrm{I}}) < 0$ is sufficient for a shear shock. There are no restrictions on $Q(\varepsilon_{\mathrm{I}})$, however. By (17) and (12), we see that $Q(\varepsilon_{\mathrm{I}})$ and V_{I}' have the same sign. Hence, (24)–(26) imply that if $Q(\varepsilon_{\mathrm{I}})$ or V_{I}' is negative [resp. positive], then ε, σ, u increase [resp. decrease] across the shear wave for sufficiently weak impacts; if $Q(\varepsilon_{\mathrm{I}})$ or V_{I}' is zero, then ε, σ, u are constant to within an error of order γ_{F}^4.

Solving $(18)_2$ for γ in terms of V shows that γ has a square root singularity at $V = V_{\mathrm{I}}$, i.e., at the wavefront; likewise so do τ and v. Since $V = X/t$, the strain rate, stress rate, and particle acceleration are infinite at the wavefront, as noted by Abou-Sayed & Clifton (3) for isotropic materials.

At the rear of the shear wave (i.e., at the final state), (29) and (30) reduce to

$$\gamma_{\mathrm{F}} \overset{5}{\approx} \left(\frac{2}{3} + \frac{1}{3}\frac{V_{\mathrm{I}}}{V_{\mathrm{F}}}\right)\frac{v_{\mathrm{F}}}{V_{\mathrm{I}}} \overset{5}{\approx} (1 + \tfrac{1}{3}\Delta_{\mathrm{S}})\,\frac{v_{\mathrm{F}}}{V_{\mathrm{I}}}, \quad (31)$$

$$\tau_{\mathrm{F}} \overset{5}{\approx} \rho_0(\tfrac{2}{3}V_{\mathrm{I}} + \tfrac{1}{3}V_{\mathrm{F}})v_{\mathrm{F}} \overset{5}{\approx} (1 - \tfrac{1}{3}\Delta_{\mathrm{S}})\rho_0 V_{\mathrm{I}} v_{\mathrm{F}}, \quad (32)$$

where $\Delta_{\mathrm{S}} \equiv V_{\mathrm{I}}/V_{\mathrm{F}} - 1 = V_{\mathrm{I}}\, t_{\mathrm{R}}(X)/X$, and $t_{\mathrm{R}}(X) = X/V_{\mathrm{F}} - X/V_{\mathrm{I}}$ is the rise time of the shear wave at X. By (22),

$$\begin{aligned}\frac{t_{\mathrm{R}}(X)}{X} &= \frac{1}{V_{\mathrm{F}}} - \frac{1}{V_{\mathrm{I}}} \overset{4}{\approx} \frac{R(\varepsilon_{\mathrm{I}})}{2V_{\mathrm{I}}^3}\left(\frac{v_{\mathrm{F}}}{V_{\mathrm{I}}}\right)^2 \\ &\approx \frac{1}{V_0}\left[\frac{6(V_0')^2}{U_0^2 - V_0^2} - \frac{h(0)}{2V_0^2}\right]\left(\frac{v_{\mathrm{F}}}{V_0}\right)^2.\end{aligned} \quad (33)$$

The top formula in (33) includes dependence of the rise time on the uniaxial strain ε_{I} ahead of the shear wave. The bottom formula neglects this dependence; it is obtained by setting $\varepsilon_{\mathrm{I}} = 0$ and using (19), and contains errors of order γ_{F}^4 and $\varepsilon_{\mathrm{I}}\gamma_{\mathrm{F}}^2$. Here U_0 and V_0 are the longitudinal and shear wave speeds in the natural state, and V_0' denotes the value of V_{I}' at $\varepsilon_{\mathrm{I}} = 0$. Thus $U_0 \& V_0$, V_0', and $h(0)$ are 2nd, 3rd, and 4th order elastic constants, respectively. An analogous formula in Abou-Sayed & Clifton (3), eqn. (46), is off by a factor of 3/2 (the result of neglecting changes in ε across the shear wave), and missing the $h(0)$ term (the result of neglecting 4th order strain terms in the internal energy).

From (22) and (19) we obtain an approximation for the 4th order coefficient h in (8) evaluated at the uniaxial strain ε_{I} ahead of the shear wave:

$$h(\varepsilon_{\mathrm{I}}) \overset{2}{\approx} \frac{12\,V_{\mathrm{I}}^2}{U_{\mathrm{I}}^2 - V_{\mathrm{I}}^2}\,(V_{\mathrm{I}}')^2 - 4V_{\mathrm{I}}^3\,\frac{V_{\mathrm{I}} - V_{\mathrm{F}}}{v_{\mathrm{F}}^2}. \quad (34)$$

V_{I}' can be estimated by differentiating the $V_{\mathrm{I}}(\varepsilon_{\mathrm{I}})$ curve obtained from a series of oblique plate impact tests, or by solving (26) and (17):

$$V_{\mathrm{I}}' \overset{2}{\approx} -\left(\frac{u_{\mathrm{F}} - u_{\mathrm{I}}}{v_{\mathrm{F}}^2}\right)(U_{\mathrm{I}}^2 - V_{\mathrm{I}}^2). \quad (35)$$

When this is substituted into (34), we obtain

$$\begin{aligned}h(\varepsilon_{\mathrm{I}}) \overset{2}{\approx} 12\left(\frac{u_{\mathrm{F}} - u_{\mathrm{I}}}{v_{\mathrm{F}}^2}\right)^2 (U_{\mathrm{I}}^2 - V_{\mathrm{I}}^2)\, V_{\mathrm{I}}^2 \\ - 4V_{\mathrm{I}}^3\,\frac{V_{\mathrm{I}} - V_{\mathrm{F}}}{v_{\mathrm{F}}^2}.\end{aligned} \quad (36)$$

The right-hand sides of (35) and (36) involve particle velocities and wave speeds only. These could be measured in a single test, at least if the longitudinal wave is simple. If the longitudinal wave is a shock with speed $\mathbb{U}$, then U_{I}^2 can be approximated by $U_{\mathrm{I}}^2 \approx 2\,\mathbb{U}^2 - U_0^2$ to within an error of order $\varepsilon_{\mathrm{I}}^2$, or calculated from $(12)_1$ if f is known.

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UNIVERSAL RELATIONS FOR PRESSURE-SHEAR WAVES IN NONLINEAR ELASTIC SOLIDS

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The centered shock waves or simple waves generated by oblique plate impact of nonlinear elastic solids are studied. Attention is restricted to materials for which the fast wave is purely longitudinal. This includes isotropic materials and appropriately aligned orthotropic or transversely isotropic materials. The emphasis is on the derivation of exact or approximate relations between stress, strain, particle velocities, and wave speeds which are *universal* in the sense that they do not involve material constants or material functions. For isotropic materials, universal relations for the difference between the longitudinal and lateral stresses are derived.

INTRODUCTION

Consider a homogeneous target plate initially at rest and stress-free. (X, Y, Z) are the Cartesian coordinates of a typical point in this natural reference state, with the X-axis normal to the target face. In an oblique plate impact test, the particle velocity $(u_{\mathsf{F}}, v_{\mathsf{F}}, 0)$ imparted to the face of the target by the flyer plate has a normal (X) component $u_{\mathsf{F}} > 0$ and a transverse (Y) component $v_{\mathsf{F}} > 0$. The impact is assumed weak enough that the response of the flyer and target is elastic. We restrict attention to points in the target interior and to times prior to the arrival of waves reflected from free surfaces or material interfaces. Then the motion consists of plane *pressure-shear* waves propagating in the X direction. For nonlinear isotropic elastic solids, this problem has been studied by Bland (1), Davison (2), and Abou-Sayed & Clifton (3). As noted by Scheidler (4), many of their results remain valid if the material is orthotropic relative to the X, Y, Z axes, or if it is transversely isotropic about either the X axis or the Z axis.

Assuming one of the above material symmetry conditions, the motion has the general form

$$x = X + \bar{d}(X,t), \;\; y = Y + \hat{d}(X,t), \;\; z = Z, \tag{1}$$

where (x, y, z) is the position at time t of the material point initially at (X, Y, Z). Let $\mathbf{P}$ and $\mathbf{T}$ denote the 1st Piola-Kirchhoff and Cauchy stress tensors. Then the normal or longitudinal components of stress and strain, taken positive in compression, are

$$\sigma_X \equiv -T_{XX} = -P_{XX}, \quad \varepsilon \equiv 1 - \frac{\partial x}{\partial X}. \tag{2}$$

The lateral stress, taken positive in compression, is

$$\sigma_Y \equiv -T_{YY} = \frac{-P_{YY} - \gamma\tau_{XY}}{1-\varepsilon}. \tag{3}$$

The shear stress and shear strain are

$$\tau_{XY} \equiv -T_{XY} = -T_{YX} = -P_{YX}, \quad \gamma \equiv -\frac{\partial y}{\partial X}. \tag{4}$$

The signs have been chosen so that ε, γ, σ_X, σ_Y, and τ_{XY}, as well as the normal and transverse components of the particle velocity, $u = \partial x/\partial t$ and $v = \partial y/\partial t$, should be nonnegative for the impact problem considered here.

Since ε and γ are the only nonzero strain components, $e = \hat{e}(\varepsilon, \gamma, s)$, where e and s are the internal energy and entropy per unit mass. Heat conduction is neglected in the analysis, so the response is isentropic except for entropy jumps

across shocks. ρ_0 denotes the density in the natural state. The material symmetry restrictions imply that e and $\sigma_X = \rho_0 \partial e/\partial\varepsilon$ are even functions of γ for fixed ε and s, whereas $\tau_{XY} = \rho_0 \partial e/\partial\gamma$ is an odd function of γ. These even/odd properties result in qualitative differences in the behavior of longitudinal and shear waves.

Two centered waves are generated on impact. The material symmetry restrictions imply that the fast wave is purely longitudinal, and so brings the material to an intermediate state of uniform uniaxial strain into which the slower shear wave propagates. The shear wave brings the material to a final state of uniform compression and shear. Quantities in the natural, intermediate, and final states are denoted by 0, I, and F subscripts, respectively. To simplify the notation, we let $\sigma = \sigma_X$ and $\tau = \tau_{XY}$. In particular, $\gamma_{\mathrm{I}} = \tau_{\mathrm{I}} = v_{\mathrm{I}} = 0$. The shear wave is often called a quasi-transverse wave since it also induces changes in the longitudinal quantities ε, σ, and u. For the classes of materials considered here, these changes are 2nd order in γ_{F}, whereas τ_{F} and v_{F} are 1st order in γ_{F}.

For sufficiently small strains, the centered waves are either shocks or simple waves, depending on the nonlinear elastic properties of the material. The case where both waves are simple is illustrated in Fig. 1.

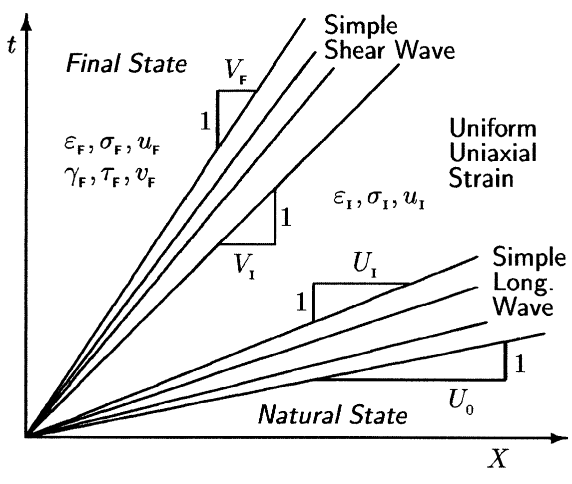

FIGURE 1. Simple waves generated by oblique plate impact. Quantities in the *intermediate* state (ahead of the shear wave) and *final* state (behind the shear wave) are denoted by I and F subscripts, respectively.

The Lagrangian acoustic (or characteristic or acceleration) longitudinal and shear wave speeds are denoted by U and V, respectively. They are even functions of γ for fixed ε and s. For the case of a simple shear wave, this implies that the time histories of γ, τ, and v have square root singularities at the shear wave front; cf. Abou-Sayed & Clifton (3) for the isotropic case and Scheidler (4) for the more general case considered here. On the other hand, for a simple longitudinal wave the time histories of ε, σ, and u are smooth (and hence approximately linear) throughout the wave, and such waves are often referred to as ramp waves. Since the analysis of the longitudinal wave is standard, the remainder of this paper focuses on the shear wave. The results are valid for arbitrary finite (elastic) longitudinal strains ε_{I} ahead of the shear wave. Due to limitations of space some proofs are omitted. For the case of a simple shear wave, derivations of some of the results can be found in Scheidler (4).

SHEAR SHOCKS

In this section we consider the case where the shear wave is a shock. The jump conditions are

$$\gamma_{\mathsf{F}} = \frac{v_{\mathsf{F}}}{\mathbb{V}}, \qquad \varepsilon_{\mathsf{F}} - \varepsilon_{\mathrm{I}} = \frac{u_{\mathsf{F}} - u_{\mathrm{I}}}{\mathbb{V}}, \tag{5}$$

$$\tau_{\mathsf{F}} = \rho_0 \mathbb{V} v_{\mathsf{F}}, \quad \sigma_{\mathsf{F}} - \sigma_{\mathrm{I}} = \rho_0 \mathbb{V}(u_{\mathsf{F}} - u_{\mathrm{I}}), \tag{6}$$

where $\mathbb{V}$ is the Lagrangian shear shock speed. These relations imply

$$\tau_{\mathsf{F}} = \rho_0 \mathbb{V}^2 \gamma_{\mathsf{F}}, \quad \sigma_{\mathsf{F}} - \sigma_{\mathrm{I}} = \rho_0 \mathbb{V}^2 (\varepsilon_{\mathsf{F}} - \varepsilon_{\mathrm{I}}). \tag{7}$$

$\mathbb{V}$, V_{F}, and V_{I} all differ by terms of order $\gamma_{\mathsf{F}}{}^2$. Stability arguments imply that $V_{\mathsf{F}} > \mathbb{V} > V_{\mathrm{I}}$, i.e., the shock wave speed is bounded below by the acoustic wave speed ahead of the shock and above by the acoustic wave speed behind the shock. $\mathbb{V}$ may be approximated by the average of these acoustic wave speeds: $\mathbb{V} = \frac{1}{2}V_{\mathrm{I}} + \frac{1}{2}V_{\mathsf{F}} + \mathcal{O}(\gamma_{\mathsf{F}}{}^2)$. Such an approximation will be abbreviated by

$$\mathbb{V} \overset{2}{\approx} \tfrac{1}{2}V_{\mathrm{I}} + \tfrac{1}{2}V_{\mathsf{F}}, \tag{8}$$

where, in general, $a \overset{n}{\approx} b$ means that $a = b$ to within an error of order $\gamma_{\mathsf{F}}{}^n$.

An interesting result, which has no analog for longitudinal shocks, is that the approximation (8) may be improved simply by changing the weighting factors:

$$\mathbb{V} \overset{4}{\approx} \tfrac{2}{3}V_{\rm I} + \tfrac{1}{3}V_{\rm F}\,. \tag{9}$$

This is a consequence of the fact that τ is an odd function of γ. (9) is equivalent to either of the relations

$$\mathbb{V} - V_{\rm I} \overset{4}{\approx} \tfrac{1}{3}\,(V_{\rm F} - V_{\rm I})\,, \tag{10}$$

$$V_{\rm F} - \mathbb{V} \overset{4}{\approx} \tfrac{2}{3}\,(V_{\rm F} - V_{\rm I})\,. \tag{11}$$

Bland(1965, p.764) obtained (10) for the special case of isotropic materials with infinitesimal uniaxial strain ahead of the shear wave. Using the fact the sum of the weighting factors in (9) is 1, it may be shown that (9)–(11) are equivalent to analogous formulas for the squares of the wave speeds:

$$\mathbb{V}^2 \overset{4}{\approx} \tfrac{2}{3}V_{\rm I}^{\,2} + \tfrac{1}{3}V_{\rm F}^{\,2}\,, \tag{12}$$

$$\mathbb{V}^2 - V_{\rm I}^{\,2} \overset{4}{\approx} \tfrac{1}{3}\,(V_{\rm F}^{\,2} - V_{\rm I}^{\,2})\,, \tag{13}$$

$$V_{\rm F}^{\,2} - \mathbb{V}^2 \overset{4}{\approx} \tfrac{2}{3}\,(V_{\rm F}^{\,2} - V_{\rm I}^{\,2})\,. \tag{14}$$

SIMPLE SHEAR WAVES

Although ε, γ, etc. vary continuously throughout a centered simple wave, it is well-known that the net changes across such a wave may be approximated by the jump conditions across a shock, with the shock speed replaced by the speed of the simple wave. In particular, for a centered simple shear wave we have the following approximate analogs of the jump conditions (5)–(7):

$$\gamma_{\rm F} \overset{3}{\approx} \frac{v_{\rm F}}{V_{\rm I}}\,, \qquad \varepsilon_{\rm F} - \varepsilon_{\rm I} \overset{4}{\approx} \frac{u_{\rm F} - u_{\rm I}}{V_{\rm I}}\,, \tag{15}$$

$$\tau_{\rm F} \overset{3}{\approx} \rho_0 V_{\rm I}\, v_{\rm F}\,, \quad \sigma_{\rm F} - \sigma_{\rm I} \overset{4}{\approx} \rho_0 V_{\rm I}(u_{\rm F} - u_{\rm I}), \tag{16}$$

$$\tau_{\rm F} \overset{3}{\approx} \rho_0 V_{\rm I}^{\,2}\gamma_{\rm F}\,, \quad \sigma_{\rm F} - \sigma_{\rm I} \overset{4}{\approx} \rho_0 V_{\rm I}^{\,2}(\varepsilon_{\rm F} - \varepsilon_{\rm I}). \tag{17}$$

In the above approximations the speed $V_{\rm I}$ of the simple wave front may be replaced by the speed $V_{\rm F}$ of rear of the wave. More accurate approximations can be obtained by replacing $V_{\rm I}$ in (15)–(17) by appropriate weighted averages of $V_{\rm I}$ and $V_{\rm F}$. For the transverse variables we have

$$\gamma_{\rm F} \overset{5}{\approx} \frac{v_{\rm F}}{\frac{2}{3}V_{\rm I} + \frac{1}{3}V_{\rm F}}\,, \tag{18}$$

$$\begin{aligned} \tau_{\rm F} &\overset{5}{\approx} \rho_0(\tfrac{2}{3}V_{\rm I} + \tfrac{1}{3}V_{\rm F})v_{\rm F} \\ &\overset{5}{\approx} \rho_0(\tfrac{2}{3}V_{\rm I} + \tfrac{1}{3}V_{\rm F})^2\gamma_{\rm F} \\ &\overset{5}{\approx} \rho_0(\tfrac{2}{3}V_{\rm I}^{\,2} + \tfrac{1}{3}V_{\rm F}^{\,2})\,\gamma_{\rm F}\,. \end{aligned} \tag{19}$$

For the longitudinal variables we have

$$\varepsilon_{\rm F} - \varepsilon_{\rm I} \overset{6}{\approx} \frac{u_{\rm F} - u_{\rm I}}{\frac{1}{2}V_{\rm I} + \frac{1}{2}V_{\rm F}}\,, \tag{20}$$

$$\begin{aligned} \sigma_{\rm F} - \sigma_{\rm I} &\overset{6}{\approx} \rho_0(\tfrac{1}{2}V_{\rm I} + \tfrac{1}{2}V_{\rm F})\,(u_{\rm F} - u_{\rm I}) \\ &\overset{6}{\approx} \rho_0(\tfrac{1}{2}V_{\rm I} + \tfrac{1}{2}V_{\rm F})^2\,(\varepsilon_{\rm F} - \varepsilon_{\rm I}) \\ &\overset{6}{\approx} \rho_0(\tfrac{1}{2}V_{\rm I}^{\,2} + \tfrac{1}{2}V_{\rm F}^{\,2})\,(\varepsilon_{\rm F} - \varepsilon_{\rm I})\,. \end{aligned} \tag{21}$$

APPROXIMATIONS FOR $\partial V_{\rm I}/\partial\varepsilon_{\rm I}$

Since $V = \hat{V}(\varepsilon, \gamma, s)$, in the intermediate uniaxially strained state where $\gamma = \gamma_{\rm I} = 0$ it follows that the acoustic shear wave speed V depends only on the uniaxial strain $\varepsilon = \varepsilon_{\rm I}$ and entropy $s = s_{\rm I}$: $V_{\rm I} = \hat{V}(\varepsilon_{\rm I}, 0, s_{\rm I})$. Then $\partial V_{\rm I}/\partial\varepsilon_{\rm I}$ is the rate of change of the acoustic shear wave speed with respect to uniaxial strain, evaluated at the intermediate uniaxially strained state. If the oblique plate impact generates a shear shock, then

$$\begin{aligned} \frac{\varepsilon_{\rm F} - \varepsilon_{\rm I}}{\gamma_{\rm F}^{\,2}} &= \rho_0\mathbb{V}^2\,\frac{\sigma_{\rm F} - \sigma_{\rm I}}{\tau_{\rm F}^{\,2}} = \mathbb{V}\,\frac{u_{\rm F} - u_{\rm I}}{v_{\rm F}^{\,2}} \\ &\overset{2}{\approx} \frac{-\mathbb{V}}{U_{\rm I}^{\,2} - \mathbb{V}^2}\,\frac{\partial V_{\rm I}}{\partial\varepsilon_{\rm I}}\,, \end{aligned} \tag{22}$$

where the top formulas follow immediately from (5)–(6), and the bottom formula can be obtained by expanding the stress quotient in powers of $\gamma_{\rm F}$. Similarly, if the shear wave is a simple wave, then

$$\begin{aligned} \frac{\varepsilon_{\rm F} - \varepsilon_{\rm I}}{\gamma_{\rm F}^{\,2}} &\overset{2}{\approx} \rho_0 V_{\rm I}^{\,2}\,\frac{\sigma_{\rm F} - \sigma_{\rm I}}{\tau_{\rm F}^{\,2}} \overset{2}{\approx} V_{\rm I}\,\frac{u_{\rm F} - u_{\rm I}}{v_{\rm F}^{\,2}} \\ &\overset{2}{\approx} \frac{-V_{\rm I}}{U_{\rm I}^{\,2} - V_{\rm I}^{\,2}}\,\frac{\partial V_{\rm I}}{\partial\varepsilon_{\rm I}}\,, \end{aligned} \tag{23}$$

where the top formulas follow from (15)–(16). In (23) the speed $V_{\rm I}$ of the simple wave front may be replaced by the speed $V_{\rm F}$ of rear of the wave. If we set

$$\mathcal{V} \equiv \begin{cases} \mathbb{V}\,, & \text{for shear shock} \\ V_{\rm I} \text{ or } V_{\rm F}\,, & \text{for simple shear wave} \end{cases} \tag{24}$$

and solve (22) or (23) for $\partial V_{\rm I}/\partial\varepsilon_{\rm I}$, we obtain

$$\frac{\partial V_{\text{I}}}{\partial \varepsilon_{\text{I}}} \overset{2}{\approx} -(U_{\text{I}}^2 - \mathcal{V}^2)(u_{\text{F}} - u_{\text{I}})/v_{\text{F}}^2$$
$$\overset{2}{\approx} -\left(\frac{U_{\text{I}}^2 - \mathcal{V}^2}{\mathcal{V}}\right)\left(\frac{\varepsilon_{\text{F}} - \varepsilon_{\text{I}}}{\gamma_{\text{F}}^2}\right) \qquad (25)$$
$$\overset{2}{\approx} -\rho_0 \mathcal{V}\,(U_{\text{I}}^2 - \mathcal{V}^2)(\sigma_{\text{F}} - \sigma_{\text{I}})/\tau_{\text{F}}^2 .$$

Simultaneous measurement of longitudinal and transverse particle velocities and wave speeds in oblique plate impact tests have been performed by Gupta and coworkers using in-material electromagnetic particle velocity gauges; cf. (5). Such measurements could be used to estimate $\partial V_{\text{I}}/\partial \varepsilon_{\text{I}}$ by means of the relation $(25)_1$.

ISOTROPIC ELASTIC SOLIDS

We return to the original notation of σ_X and τ_{XY} for the normal and shear stresses. Since the lateral stress σ_Y does not appear in the momentum balance equations, the wave analysis yields no information on σ_Y, at least for the anisotropic solids considered up to this point. However, isotropy places restrictions on σ_Y, as we now show.

$\boldsymbol{F} = \partial \boldsymbol{x}/\partial \boldsymbol{X}$ is the deformation gradient. For an isotropic elastic solid, the Cauchy stress tensor $\boldsymbol{T}$ commutes with $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$: $\boldsymbol{TB} = \boldsymbol{BT}$. For the pressure-shear motion (1), the only nontrivial restriction that follows from this result is $T_{XY}(B_{XX} - B_{YY}) = B_{XY}(T_{XX} - T_{YY})$. For $\gamma = 0$ (i.e., uniaxial strain) this reduces to $0 = 0$, but for $\gamma \neq 0$ it implies the universal relations

$$\sigma_X - \sigma_Y = \frac{2\varepsilon - \varepsilon^2 + \gamma^2}{1-\varepsilon}\,\frac{\tau_{XY}}{\gamma} = \frac{2\varepsilon - \varepsilon^2}{1-\varepsilon}\,\frac{\tau_{XY}}{\gamma} + \frac{\gamma\,\tau_{XY}}{1-\varepsilon} . \qquad (26)$$

If the oblique plate impact generates a shear shock, then evaluating (26) in the final state and using the jump conditions $(5)_1$–$(7)_1$ gives

$$(\sigma_X - \sigma_Y)_{\text{F}} = \frac{2\varepsilon_{\text{F}} - \varepsilon_{\text{F}}^2 + \gamma_{\text{F}}^2}{1-\varepsilon_{\text{F}}}\,\rho_0 \mathbb{V}^2 = \frac{2\varepsilon_{\text{F}} - \varepsilon_{\text{F}}^2}{1-\varepsilon_{\text{F}}}\,\rho_0 \mathbb{V}^2 + \frac{\rho_0 v_{\text{F}}^2}{1-\varepsilon_{\text{F}}} . \qquad (27)$$

In the weak shock limit γ_{F} & $v_{\text{F}} \to 0$, $\varepsilon_{\text{F}} \to \varepsilon_{\text{I}}$, and $\mathbb{V} \to V_{\text{I}}$, yielding a universal relation for the difference between the normal and lateral stresses in the intermediate uniaxially strained state ahead of the shear shock:

$$\sigma_X - \sigma_Y = \frac{2\varepsilon - \varepsilon^2}{1-\varepsilon}\,\rho_0 V^2 = \left[\left(\frac{\rho}{\rho_0}\right)^2 - 1\right]\rho \widetilde{V}^2 . \qquad (28)$$

For a simple shear wave, a similar analysis using $(15)_1$–$(17)_1$ also yields (28). Here and below, we drop the I subscript since all quantities are evaluated in the intermediate state. $(28)_2$ was first derived by Scheidler (6) by a different method; $\widetilde{V} = (1-\varepsilon)V$ is the Eulerian acoustic shear wave speed and ρ the density. (28) also yields formulas for the pressure p and maximum shear stress τ_{max} in the uniaxially strained state, since

$$2\,\tau_{\mathsf{max}} = \sigma_X - \sigma_Y\,, \quad p = \sigma_X - \tfrac{4}{3}\,\tau_{\mathsf{max}}\,. \qquad (29)$$

From (28)–(29) and $\rho_0 U^2 = d\sigma_X/d\varepsilon$, we obtain

$$\frac{d\tau_{\mathsf{max}}}{d\varepsilon} = \rho_0 V^2 + \mathcal{A}(\varepsilon)\,\tau_{\mathsf{max}}\,, \qquad (30)$$

$$\frac{dp}{d\varepsilon} = (\rho_0 U^2 - \tfrac{4}{3}\rho_0 V^2) - \tfrac{4}{3}\mathcal{A}(\varepsilon)\tau_{\mathsf{max}}\,, \qquad (31)$$

$$\mathcal{A}(\varepsilon) = \frac{1}{1-\varepsilon} + \frac{2}{V}\frac{\partial V}{\partial \varepsilon} . \qquad (32)$$

Since approximations for $\partial V/\partial \varepsilon$ are given by (25), for isotropic solids a single oblique plate impact test with in-material particle velocity gauges can yield estimates for σ_Y, τ_{max}, and p in the uniaxially strained state ahead of the shear wave, as well as for their derivatives w.r.t. uniaxial strain.

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APPROXIMATE UNIVERSAL RELATIONS BETWEEN SHOCK AND ACCELERATION WAVE SPEEDS FOR OBLIQUE PLATE IMPACT OF INELASTIC SOLIDS

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Abstract. We derive some approximate relations between shear wave speeds in inelastic solids subject to oblique plate impact. Attention is restricted to materials for which the fast wave is purely longitudinal. This includes isotropic solids as well as appropriately aligned orthotropic solids. For the case where the slower wave is a shear shock, we obtain approximate relations between this shock speed and the shear sound speeds (i.e., acceleration wave speeds) immediately ahead of and behind the shock. These relations are universally valid for isotropic or orthotropic elastic solids as well as inelastic solids with instantaneous elastic response.
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INTRODUCTION

The waves studied in this paper are of the type generated in an oblique plate impact test. A flyer plate impacts a parallel target plate, with both plates inclined to the direction of motion of the flyer. The particle velocity imparted to the target plate has components normal and parallel to the impact surface. We choose a Cartesian coordinate system with the X_1 and X_2 axes in these two directions, respectively, and restrict attention to points in the target interior and to times prior to the arrival of reflected waves from free surfaces or material interfaces. If the material is homogeneous the impact generates plane waves propagating in the X_1 direction: a quasi-longitudinal wave propagates into undeformed material at rest, followed by a slower moving quasi-transverse (i.e., shear) wave. Here *quasi* refers to the fact that the longitudinal and transverse waves are generally accompanied by relatively smaller changes in transverse and longitudinal quantities, respectively. If the material is isotropic, however, the fast wave will be purely longitudinal. In this case the shear wave propagates into uniaxially strained material, a condition essential for our main results. But even for isotropic solids this shear wave will still be quasi-transverse due to nonlinearities in the material response; cf. Bland [1], Davison [2], Abou-Sayed & Clifton [3], and Gupta [4] for some of the earlier theoretical and experimental work on this problem.

Depending on the materials and the strength of the impact, the waves generated could be shock or ramp (i.e., continuous) waves. We consider only the case where the shear wave is a shock; whether the uniaxially strained state ahead of this shear shock is generated by a shock or ramp wave is inconsequential.

Bland (p. 764 of [1]) showed that *to within an error of fourth order in the jump in shear strain across a shear shock, the shock speed exceeds the sound speed ahead of it by one third the excess of the sound speed behind it over the sound speed ahead of it*. Of course, the sound speeds refer to shear waves as well. Bland derived this result for homogeneous nonlinear isotropic hyperelastic solids, assuming the shear shock is “propagating into an unstrained medium at rest.” But even purely transverse loadings on the boundary would generate a longitudinal precursor, so his restriction is not experimentally achievable.

In [5] I noted that Bland's approximate relation holds with no restriction on the uniaxial strain ahead of the shear shock, and that "isotropic" could be weakened to "orthotropic relative to the coordinate axes." The proof (which was omitted) utilized results in an earlier paper [6], making substantial use of the hyperelasticity assumption (i.e., the internal energy is a potential for the stress) and also of the method of characteristics for calculating the sound speeds.

Here I show that Bland's approximate relation also holds for a large class of inelastic solids with properties that may vary smoothly in the X_1 direction. The proof makes no use of a potential relation for the stress tensor.[1] The method of characteristics for calculating sound speeds must be abandoned since at this level of generality the equations of motion are not quasi-linear hyperbolic PDEs. Instead, we use the theory of singular surfaces since the sound speeds may be identified with acceleration wave speeds.[2]

INSTANTANEOUS ELASTIC RESPONSE

The results in this section are stated for an arbitrary motion $\mathbf{x} = \widehat{\mathbf{x}}(\mathbf{X},t)$ relative to an arbitrary Cartesian coordinate system: $\mathbf{x} = (x_1,x_2,x_3)$ is the position at time t of the material point initially at $\mathbf{X} = (X_1,X_2,X_3)$.The deformation gradient $\mathbf{F}$ has components $F_{ij} = \partial x_i/\partial X_j$. Let $\mathbf{T}$ and $\mathbf{S} = (\det\mathbf{F})\mathbf{T}\mathbf{F}^{-T}$ denote the Cauchy and first Piola-Kirchhoff stress tensors. The value of the stress at time t can depend not only on $\mathbf{F}(t)$, as for an elastic material, but also on the entire past history of the deformation gradient:

$$\mathbf{S}(t) = \widehat{\mathbf{S}}(\mathbf{F}(t);\mathbf{F}^t)\,. \tag{1}$$

Here $\mathbf{F}^t$ is the history of $\mathbf{F}$ up to time t, the (tensor-valued) function defined on the positive reals by $\mathbf{F}^t(s) \equiv \mathbf{F}(t-s)$ for $s > 0$. Thus $\widehat{\mathbf{S}}$ is a function of $\mathbf{F}(t)$ and a functional of $\mathbf{F}^t$.[3] For any orthogonal tensor $\mathbf{Q}$ and any tensor $\mathbf{H}$ in the material symmetry group,[4]

$$\widehat{\mathbf{S}}(\mathbf{QF}(t)\mathbf{H};\mathbf{QF}^t\mathbf{H}) = \mathbf{Q}\widehat{\mathbf{S}}(\mathbf{F}(t);\mathbf{F}^t)\mathbf{H}\,. \tag{2}$$

We assume that $\mathbf{S}$ is continuous at any instant t at which $\mathbf{F}$ is continuous. Across a shock at time t, we assume that

$$\mathbf{S}^- = \widehat{\mathbf{S}}(\mathbf{F}^-;\mathbf{F}^t)\,, \quad \mathbf{S}^+ = \widehat{\mathbf{S}}(\mathbf{F}^+;\mathbf{F}^t)\,, \tag{3}$$

which requires a continuous dependence on the past history across the shock front. At any instant t when $\mathbf{F}$ and its material time derivative $\dot{\mathbf{F}}$ are continuous, we assume that $\dot{\mathbf{S}}$ is continuous as well and given by

$$\dot{\mathbf{S}}(t) = \partial_1\widehat{\mathbf{S}}(\mathbf{F}(t);\mathbf{F}^t)\dot{\mathbf{F}}(t) + \widehat{\mathbf{G}}(\mathbf{F}(t);\mathbf{F}^t)\,. \tag{4}$$

The 4th-order tensor $\partial_1\widehat{\mathbf{S}}$ is the derivative of $\widehat{\mathbf{S}}$ with respect to its first argument $\mathbf{F}(t)$, holding $\mathbf{F}^t$ fixed. The functions $t \mapsto \partial_1\widehat{\mathbf{S}}(\mathbf{F}(t);\mathbf{F}^t)$ and $t \mapsto \widehat{\mathbf{G}}(\mathbf{F}(t);\mathbf{F}^t)$ are assumed to be continuous at any instant at which $\mathbf{F}$ is continuous, so that across an acceleration wave

$$[\![\dot{\mathbf{S}}]\!] = \partial_1\widehat{\mathbf{S}}(\mathbf{F}(t);\mathbf{F}^t)\,[\![\dot{\mathbf{F}}]\!]\,, \tag{5}$$

analogous to the case for an elastic material (where the dependence on $\mathbf{F}^t$ would be absent). Materials with the properties described above will be said to have *instantaneous elastic response*.[5]

Let $\mathbf{a} = \ddot{\mathbf{x}}$ denote the particle acceleration. Across an acceleration wave, balance of momentum and kinematic compatibility yield the relations

$$[\![\dot{\mathbf{S}}]\!]\mathbf{n} = -\rho_0 c[\![\mathbf{a}]\!]\,, \quad c[\![\dot{\mathbf{F}}]\!] = -[\![\mathbf{a}]\!]\otimes\mathbf{n}\,, \tag{6}$$

where $\mathbf{n}$ is the unit normal to the referential wave front, c the referential (or Lagrangean) wave speed, and ρ_0 the initial density. On applying (5) to $\mathbf{n}$ and using (6), we obtain the propagation condition

$$\rho_0 c^2[\![\mathbf{a}]\!] = \mathbf{A}(\mathbf{n})[\![\mathbf{a}]\!]\,. \tag{7}$$

For a given direction of propagation $\mathbf{n}$, the 2nd-order

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[1] For the materials considered here such a relation would follow from the usual form of the 2nd law of thermodynamics; cf.[7] for materials with fading memory and [8] for viscoplastic materials. For simplicity, thermodynamic effects are neglected.

[2] *Acceleration waves* are propagating singular surfaces across which the stress, strain and particle velocity are continuous but suffer jump discontinuities in their spatial gradients and time derivatives; they can be the fronts of ramp waves. Similarly, we use the term *shock wave* only for propagating singular surfaces across which the stress, strain and particle velocity suffer jump discontinuities. The jump in a quantity Ψ across a singular surface is denoted by $[\![\Psi]\!] = \Psi^- - \Psi^+$, where Ψ^- and Ψ^+ denote limits of Ψ from behind and ahead of the wave front, respectively.

[3] The results in this section are unaltered if we also allow the material properties to depend smoothly on the position $\mathbf{X}$.

[4] Indeed, $\mathbf{T}$ is unchanged and $\mathbf{S}$ changes to $\mathbf{SH}$ when $\mathbf{F}$ is replaced by $\mathbf{FH}$ for all times. And frame-indifference implies that $\mathbf{T}$ changes to $\mathbf{QTQ}^T$ and $\mathbf{S}$ to $\mathbf{QS}$ when $\mathbf{F}$ is replaced by $\mathbf{QF}$ for all times. On combining these results, we obtain (2).

[5] These properties are possessed by materials with fading memory [7] and also by many internal state variable models and models for viscoplastic response [8].

tensor $\mathbf{A}(\mathbf{n})$, called the *acoustic tensor*, is defined by

$$\mathbf{A}(\mathbf{n})\mathbf{b} = \Big(\partial_1 \widehat{\mathbf{S}}(\mathbf{F}(t);\mathbf{F}^t)\big[\mathbf{b}\otimes\mathbf{n}\big]\Big)\,\mathbf{n} \tag{8}$$

for any vector $\mathbf{b}$. Thus (7) states that $[\![\mathbf{a}]\!]$ must be an eigenvector of the acoustic tensor and that $\rho_0 c^2$ is its eigenvalue. The ij component of $\mathbf{A}(\mathbf{n})$ is given by $\sum_{k,l=1}^{3}(\partial S_{ik}/\partial F_{jl})n_k n_l$, holding $\mathbf{F}^t$ fixed. When $\mathbf{n}$ is the unit vector $\mathbf{e}_1$ along the X_1 axis, this reduces to

$$\mathbf{A}(\mathbf{e}_1) = \begin{bmatrix} \dfrac{\partial S_{11}}{\partial F_{11}} & \dfrac{\partial S_{11}}{\partial F_{21}} & \dfrac{\partial S_{11}}{\partial F_{31}} \\ \dfrac{\partial S_{21}}{\partial F_{11}} & \dfrac{\partial S_{21}}{\partial F_{21}} & \dfrac{\partial S_{21}}{\partial F_{31}} \\ \dfrac{\partial S_{31}}{\partial F_{11}} & \dfrac{\partial S_{31}}{\partial F_{21}} & \dfrac{\partial S_{31}}{\partial F_{31}} \end{bmatrix}. \tag{9}$$

PLANE WAVES

Assume the displacements depend on X_1 and t only:

$$x_i - X_i = \tilde{x}_i(X_1,t)\,, \quad i = 1,2,3\,. \tag{10}$$

Then the same holds for $\mathbf{F}$, which reduces to

$$\mathbf{F}(X_1,t) = \begin{bmatrix} F_{11} & 0 & 0 \\ F_{21} & 1 & 0 \\ F_{31} & 0 & 1 \end{bmatrix}. \tag{11}$$

For the constitutive relation (1), $\mathbf{S}$ also depends only on X_1 and t. For fixed X_1 we may write this as

$$S_{ij}(t) = \widehat{S}_{ij}(F_{11}(t),F_{21}(t),F_{31}(t);\,F_{11}^t,F_{21}^t,F_{31}^t). \tag{12}$$

Momentum balance reduces to

$$\frac{\partial S_{i1}}{\partial X_1} = \rho_0 a_i\,, \quad i = 1,2,3\,, \tag{13}$$

which involves only the stress components S_{11}, S_{21} and S_{31}. This continues to hold if the material properties vary smoothly with X_1. The motion is a plane wave propagating in the X_1 direction.

Now assume the symmetry group contains a 180° rotation about the X_3 axis; its matrix is given by $\mathrm{diag}(-1,-1,1)$. On choosing this for $\mathbf{Q}$ and $\mathbf{H}$ in (2), we obtain the following restrictions on (12):

$$\begin{aligned} &\widehat{S}_{ij}(F_{11},F_{21},-F_{31};\,F_{11}^t,F_{21}^t,-F_{31}^t) \\ &\quad = \pm\widehat{S}_{ij}(F_{11},F_{21},F_{31};\,F_{11}^t,F_{21}^t,F_{31}^t)\,, \end{aligned} \tag{14}$$

where the $+$ case holds for $ij = 11,22,33,12,21$, and the $-$ case for $ij = 13,23,32,31$. For the latter case we see that $S_{ij} = -S_{ij}$ when $F_{31} \equiv 0$. Thus $\mathbf{S}$ reduces to

$$\mathbf{S}(X_1,t) = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{33} \end{bmatrix} \tag{15}$$

when $F_{31} \equiv 0$. In this case we also have

$$\left.\frac{\partial S_{31}}{\partial F_{11}}\right|_{F_{31}\equiv 0} = \left.\frac{\partial S_{31}}{\partial F_{21}}\right|_{F_{31}\equiv 0} = 0\,. \tag{16}$$

Now assume that $x_3 \equiv X_3$, which is consistent with the boundary conditions discussed in the Introduction. Then $F_{31} \equiv 0$ and $a_3 \equiv 0$, so the 3rd component of the momentum balance equation (13) is satisfied since $S_{31} \equiv 0$. By (14) we see that S_{11} and S_{21} are even functions of F_{31} when $F_{31}^t = 0$. Thus we also have

$$\left.\frac{\partial S_{11}}{\partial F_{31}}\right|_{F_{31}\equiv 0} = \left.\frac{\partial S_{21}}{\partial F_{31}}\right|_{F_{31}\equiv 0} = 0\,. \tag{17}$$

Define the longitudinal stress and strain, σ and ε, and the shear stress and strain, τ and γ, by

$$\begin{aligned} \sigma &= -S_{11} = -T_{11}\,, & \tau &= -S_{21} = -T_{21}\,, \\ \varepsilon &= 1 - \frac{\partial x_1}{\partial X_1} = 1 - F_{11}\,, & \gamma &= -\frac{\partial x_2}{\partial X_1} = -F_{21}\,. \end{aligned} \tag{18}$$

These variables should be nonnegative for the impact problem considered here, and we may now write

$$\begin{Bmatrix} \sigma(t) \\ \tau(t) \end{Bmatrix} = \begin{Bmatrix} \widehat{\sigma} \\ \widehat{\tau} \end{Bmatrix}\big(\varepsilon(t),\gamma(t);\,\varepsilon^t,\gamma^t\big)\,. \tag{19}$$

By (16)–(18), the acoustic tensor (9) for waves propagating in the X_1 direction reduces to

$$\mathbf{A}(\mathbf{e}_1) = \begin{bmatrix} \dfrac{\partial\sigma}{\partial\varepsilon} & \dfrac{\partial\sigma}{\partial\gamma} & 0 \\ \dfrac{\partial\tau}{\partial\varepsilon} & \dfrac{\partial\tau}{\partial\gamma} & 0 \\ 0 & 0 & \left.\dfrac{\partial S_{31}}{\partial F_{31}}\right|_{F_{31}\equiv 0} \end{bmatrix}. \tag{20}$$

Since, by (7), $\rho_0 c^2$ is an eigenvalue of $\mathbf{A}(\mathbf{e}_1)$, two of the Lagrangean sound (acceleration wave) speeds are obtained by equating $2\rho_0 c^2$ with

$$\frac{\partial\sigma}{\partial\varepsilon} + \frac{\partial\tau}{\partial\gamma} \pm \sqrt{\left(\frac{\partial\sigma}{\partial\varepsilon} - \frac{\partial\tau}{\partial\gamma}\right)^2 + 4\,\frac{\partial\sigma}{\partial\gamma}\,\frac{\partial\tau}{\partial\varepsilon}}\,. \tag{21}$$

The faster (quasi-longitudinal) sound speed is denoted by U, and the slower (quasi-transverse) sound

speed by V.[6]

Now assume the symmetry group also contains a 180° rotation about the X_1 axis, so that the material is orthotropic. On taking $\mathbf{Q} = \mathbf{H} = \text{diag}(1,-1,-1)$ in (2), we obtain restrictions on $\widehat{S}_{ij}$ in (12) which, in view of $F_{31} \equiv 0$ and (18)–(19), imply that for any strain values ε^* and γ^* and strain histories ε^t and γ^t,

$$\begin{aligned} \widehat{\sigma}(\varepsilon^*,-\gamma^*;\varepsilon^t,-\gamma^t) &= \widehat{\sigma}(\varepsilon^*,\gamma^*;\varepsilon^t,\gamma^t)\,, \\ \widehat{\tau}(\varepsilon^*,-\gamma^*;\varepsilon^t,-\gamma^t) &= -\widehat{\tau}(\varepsilon^*,\gamma^*;\varepsilon^t,\gamma^t)\,. \end{aligned} \tag{22}$$

It follows that if $\gamma = 0$ up to and including time t, then so are τ, $\partial\tau/\partial\varepsilon$ and $\partial\sigma/\partial\gamma$.[7]

SHEAR SHOCKS

Consider a shear shock propagating into uniaxially strained material behind the longitudinal wave in an oblique plate impact test. Let $\mathbb{V}$ denote the Lagrangean shear shock speed. The jump conditions are

$$[\![\tau]\!] = \rho_0 \mathbb{V}^2 [\![\gamma]\!]\,, \qquad [\![\sigma]\!] = \rho_0 \mathbb{V}^2 [\![\varepsilon]\!]\,; \tag{23}$$

hence

$$[\![\tau]\!][\![\varepsilon]\!] = [\![\sigma]\!][\![\gamma]\!]\,. \tag{24}$$

Since γ and τ are zero ahead of the shock, we have $\gamma^+ = \tau^+ = 0$. Thus $[\![\gamma]\!] = \gamma^-$, and by (22) the longitudinal and shear stress behind the shock are

$$\begin{aligned} \sigma^- &= \widehat{\sigma}(\varepsilon^-,[\![\gamma]\!];\varepsilon^t,0)\,, \\ [\![\tau]\!] = \tau^- &= \widehat{\tau}(\varepsilon^-,[\![\gamma]\!];\varepsilon^t,0)\,, \end{aligned} \tag{25}$$

these being even and odd functions, respectively, of $[\![\gamma]\!]$ for fixed ε^- and ε^t. But ε^- and $[\![\gamma]\!]$ are coupled through (24)–(25), which can be used to show that $[\![\varepsilon]\!]$ is of order $[\![\gamma]\!]^2$. Then $(23)_1$ and the sound speed relations imply $\mathbb{V} \approx V_+ \approx V_-$ to within an error of order $[\![\gamma]\!]^2$, where V_- and V_+ are the limiting values of the shear sound speeds from behind and ahead of the shear shock. It follows that $\mathbb{V} \approx \alpha V_+ + \beta V_-$ for any α and β satisfying $\alpha + \beta = 1$, also to within an error of order $[\![\gamma]\!]^2$.

It turns out there is an optimal choice for α and β above. From (21) and the results in this section, it is not hard to show that to an within error of order $[\![\gamma]\!]^4$,

$$\rho_0 V_-^2 \approx \frac{\partial\tau}{\partial\gamma} - \frac{\partial\sigma}{\partial\gamma}\frac{\partial\tau}{\partial\varepsilon} \Big/ \left(\frac{\partial\sigma}{\partial\varepsilon} - \frac{\partial\tau}{\partial\gamma}\right), \tag{26}$$

where the right side is evaluated behind the shock, at $(\varepsilon^-,[\![\gamma]\!];\varepsilon^t,0)$. Some lengthy calculations reveal that this equals $\rho_0(3\mathbb{V}^2 - 2V_+^2)$ to within an error of order $[\![\gamma]\!]^4$. Hence, we obtain the equivalent relations below, all valid to within an error of order $[\![\gamma]\!]^4$:

$$\begin{aligned} \mathbb{V}^2 &\approx \tfrac{2}{3}V_+^2 + \tfrac{1}{3}V_-^2 \qquad & \mathbb{V}^2 - V_+^2 &\approx \tfrac{1}{3}(V_-^2 - V_+^2) \\ V_-^2 &\approx 3\mathbb{V}^2 - 2V_+^2 \qquad & V_-^2 - \mathbb{V}^2 &\approx 2(\mathbb{V}^2 - V_+^2)\,. \end{aligned} \tag{27}$$

These yield analogous approximations for the speeds themselves to within an error of order $[\![\gamma]\!]^4$:

$$\begin{aligned} \mathbb{V} &\approx \tfrac{2}{3}V_+ + \tfrac{1}{3}V_- \qquad & \mathbb{V} - V_+ &\approx \tfrac{1}{3}(V_- - V_+) \\ V_- &\approx 3\mathbb{V} - 2V_+ \qquad & V_- - \mathbb{V} &\approx 2(\mathbb{V} - V_+) \end{aligned} \tag{28}$$

These results depend in an essential way on the condition of uniaxial strain ahead of the shear shock. Although we have neglected thermodynamic effects, it can be shown that their inclusion does not alter the above results if we make the typical assumption that the heat flux does not jump across a shock.

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[6] Note that these sound speeds are well-defined at each point and time, regardless of whether or not the corresponding acceleration wave is actually generated in the impact test. The third speed, given by $\rho_0 c^2 = (\partial S_{31}/\partial F_{31})|_{F_{31}\equiv 0}$, corresponds to a shear wave polarized along the X_3 axis and is not relevant here.

[7] Since $S_{21} = -\tau$, the fact that $\tau = 0$ for a pure uniaxial strain history suffices for the fast wave in an oblique plate impact test to be purely longitudinal. The zero derivatives imply that the relations for the longitudinal and shear sound speeds in uniaxially strained material reduce to $\rho_0 U^2 = \partial\sigma/\partial\varepsilon$ and $\rho_0 V^2 = \partial\tau/\partial\gamma$, evaluated at $\gamma \equiv 0$; since (20) is diagonal, these are pure modes.

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